Thus, for material of given composition and fixed κ, $\tau_{1/2}$ will change with the square of the radial distance within the spherical object, with a sensitivity inversely proportional to the thermal diffusivity. For a given value of r, $\tau_{1/2}$ will depend only on κ and a, again in an expected manner: for large a the half-heating times become large, also in a quadratic manner.

A quick qualitative sketch of the thermal gradients can be made if one recognizes that $\tau_{1/4}$ and $\tau_{3/4}$ are given by

$$\tau_{1/4} = (a^2/\kappa\pi^2)\{\ln(8/3) - [(\pi\sigma)^2/6]\} \tag{15}$$

$$\tau_{3/4} = (a^2/\kappa\pi^2)\{\ln 8 - [(\pi\sigma)^2/6]\} \tag{16}$$

In conclusion, the concept of the half-heating time offers a simple manner of estimating the rate of appearance of temperature distributions and the manner in which they depend on the physical properties of the system. For spherical objects, it is approximately given by a remarkably simply equation whose accuracy extends over a wide range of conditions.

Acknowledgments

The author wishes to thank the Vernon W. Krieble Foundation and the Department of Chemistry for their generous support.

References

[1] Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd ed., Oxford Univ. Press, London, 1959, p. 233.

[2] Mohsenin, N. N., *Thermal Properties of Foods and Agricultural Materials*, Gordon and Breach, New York, 1980, p. 13.

[3] Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, New York, 1980, p. 45.

[4] Grosser, A. E., "Computer Simulation of the Chemical Kinetics of Flavor Compounds in Heated Foods," *Food Science and Human Nutrition*, Elsevier, Amsterdam, 1992, pp. 123–130.

Analytical Solution for Thermal Runaway in the Surface Heating of Plates

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Nomenclature

$C(\lambda)$	=	defined in Eqs. (13) and (15)
c	=	plate thickness or half thickness, Eqs. (3) and (10)
h	=	convective coefficient
k	=	thermal conductivity
q	=	absorbed incident surface heat flux
q_i	=	incident surface heat flux
T	=	temperature
t	=	time
x, y, z	=	spatial coordinates
α	=	constant, Eq. (1)
β	=	real constant defined in Eq. (8)
κ	=	thermal diffusivity
λ	=	indexed eigenvalues
ρ	=	surface absorptance
ρ_0	=	constant surface absorptance
$\phi(x)$	=	eigenfunction

Received March 11, 1992; revision received July 6, 1992; accepted for publication July 7, 1992. This paper is declared a work of the U.S. Government and is not subject to copyright protection in the United States.

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Introduction

THERE are many instances in high energy systems in which the input heat flux to a surface is dependent on the temperature of the surface. One example is the laser heating of a metal surface in which the surface absorptance increases linearly with surface temperature.[1,2] Another example is linear particle accelerators in which the resistive losses in the cavity are proportional to the cavity surface temperature. The temperature solutions to such problems can be unstable in the sense that the surface temperature increases exponentially with time. In many cases this situation is undesirable and the rear surface (the surface opposite the heating) is cooled convectively. An analytical solution will be developed for the case of an insulated rear surface and for a convectively cooled rear surface. The stability of these temperature solutions will be examined.

Problem

A simple problem to illustrate the solution technique will now be described. A laser irradiates a metal plate that extends far into the y and z directions so that all the temperature variation is in the x direction, i.e., the problem is one-dimensional. The plate is insulated on the rear surface (an excellent approximation for short time high energy laser heating of a thin plate) as shown in Fig. 1.

The front surface absorptance is given by:

$$\rho = \rho_0 + \alpha T \tag{1}$$

The fact that the absorptance is given, to an excellent approximation, by Eq. (1) is well documented.[2] The heat flux is then given by

$$q = \rho q_i = \rho_0 q_i + \alpha q_i T \tag{2}$$

The governing equation and boundary conditions are then given by

$$\begin{aligned} &\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \\ &T(x, 0) = 0 \quad \text{for} \quad 0 < x < c \\ &-k\frac{\partial T}{\partial x} = \rho q_i = \rho_0 q_i + \alpha q_i T \quad @ \quad x = 0 \\ &-k\frac{\partial T}{\partial x} = 0 \quad @ \quad x = c \end{aligned} \tag{3}$$

The solution to Eq. (3), using the method described in the Appendix is

$$T(x, t) = -\frac{\rho_0}{\alpha} - \frac{\rho_0 q_i}{k}\sum_{m=0}^{\infty}\frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2} \cdot \exp(-\kappa\lambda_m^2 t) \tag{4}$$

Where the various terms are defined in the Appendix. The eigenfunction is

$$\phi_m = \cos\lambda_m(x - c) \tag{5}$$

where λ_m is a root of

$$\lambda_m c\tan(\lambda_m c) = -(c\alpha q_i/k) \tag{6}$$

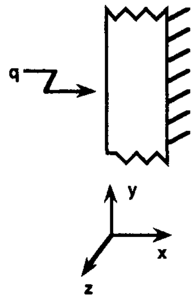

Fig. 1 Problem boundary conditions.

In solving Eq. (5), it is noted that when the eigenvalues, λ_m^2, are arranged in the following sequence:

$$\lambda_0^2 < \lambda_1^2 < \cdots < \lambda_m^2 \tag{7}$$

then the mth eigenfunction will have m zeroes on the problem interval $(0 < x < c)$.[3] Since the imaginary eigenvalues will always occupy the lowest positions in the sequence, the number of imaginary eigenvalues that exist can be determined by counting the zeroes, on the 0 to c interval, of the eigenfunction that corresponds to the first real eigenvalue. In this case, the first real eigenvalue satisfies the following inequality:

$$(\pi/2) < \lambda_1 c < \pi \tag{8}$$

Thus, Eq. (5), by inspection, has one zero on the interval $0 < x < c$ and, consequently, there is one imaginary eigenvalue and the first term of the summation in Eq. (4) is

$$-(\rho_0 q_i/k)[C(\beta)\phi_0(x)\cosh(\beta c)/\beta^2]\exp(\kappa\beta^2 t)$$

and β is real and determined from

$$\beta c \tanh(\beta c) = (c\alpha q_i/k) \tag{9}$$

where

$$\beta = (\lambda_0/i)$$

The solution [Eq. (4)] clearly contains a term that grows in time, and therefore, the temperature solution increases exponentially in time.

Additional Problems

Referring to Fig. 1, let the rear insulated surface now be convectively cooled. This situation is encountered when the heat loading on laser mirrors or particle accelerators is substantial enough to require active cooling. A stable temperature solution is one that is bounded in time. A stable temperature solution will ensure that the accelerator or mirror stabilizes at some (unspecified) temperature while an unstable solution will result in a temperature that increases exponentially in time.

Assuming the case of a laser mirror, the governing equations are

$$\begin{aligned}
&\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \\
&T(x, 0) = 0 \quad \text{for} \quad -c < x < c \\
&-k\frac{\partial T}{\partial x} = \rho q_i = \rho_0 q_i + \alpha q_i T(-c, t) \\
&-k\frac{\partial T}{\partial x} = hT(c, t)
\end{aligned} \tag{10}$$

The eigenfunctions are

$$\phi_m(x) = \cos[\lambda_m(x - c)] - (h/k\lambda_m)\sin[\lambda_m(x - c)] \tag{11}$$

and the eigenvalues are the roots of

$$\tan(2\lambda_m c) = \frac{h - q_i\alpha}{(hq_i\alpha/k\lambda_m) + k\lambda_m} \tag{12}$$

The solution, employing the same techniques discussed in the Appendix, is

$$T(x, t) = \frac{\rho_0 q_i}{k[(q_i\alpha/h) - 1] + 2q_i\alpha c}[x - c - (k/h)] - \frac{\rho_0 q_i}{k}\sum_{m=0}^{\infty}\frac{C(\lambda_m)\phi_m(x)\phi_m(-c)}{\lambda_m^2}\exp(-\kappa\lambda_m^2 t)$$

where

$$\frac{1}{C(\lambda)} = \frac{\sin(4\lambda c)}{4\lambda} + c + \frac{h^2}{k^2\lambda^2}\left[c - \frac{\sin(4\lambda c)}{4\lambda}\right] + \frac{h}{2k\lambda^2}[1 - \cos(4\lambda c)] \tag{13}$$

The interesting part of the above solution is that for the solution to be stable, the following inequality must be satisfied:

$$k(h - q_i\alpha) > 2q_i\alpha hc \tag{14}$$

otherwise, using the same reasoning as in the previous section, the first two eigenvalues will be imaginary and the temperature solution will increase without bound. Expression (14) is a convenient expression for the quality of the active cooling. If one operates comfortably within the criteria of Eq. (14), then, in principle, the mirror/accelerator will always stabilize at a specific temperature, provided that all the other boundary condition and linearity criteria are met.

Summary

An analytical solution to the high energy heating of metals in which the surface heat flux increases with temperature has been developed. The solutions result in purely imaginary eigenvalues, which lead to temperature solutions that can increase exponentially with time. For many situations, an unbounded temperature solution is undesirable. A simple expression for avoiding such a temperature solution was derived.

Appendix

There are several techniques available for solving Eq. (3), the classical product solution method being one. A technique, described by Ölcer[4] was selected, however, due to its power and the ease of extending the solution to more complex problems. Ölcer's method does have to be modified slightly to solve Eq. (3) and what follows will describe that modification.

Ölcer's method restricts αq_i to negative values. If that restriction is ignored the following expression follows from Eq. (3):

$$T(x, t) = \frac{\rho_0 q_i}{k}\sum_{m=0}^{\infty}\frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2}[1 - \exp(-\kappa\lambda_m^2 t)] \tag{A1}$$

where

$$[1/C(\lambda)] = [\sin(2\lambda c/4\lambda] + (c/2)$$

The eigenvalues can now take on imaginary values, however the functions ϕ_m still comprise a complete orthogonal set.[3] Although Eq. (A1) is apparently a solution to Eq. (3), it has two distinct disadvantages. It diverges slowly and, because its derivative with respect to x does not converge uniformly, it cannot be proven to satisfy the boundary conditions in Eq. (3).

The following technique is, therefore, employed. Take the limit of Eq. (A1) with respect to time:

$$\lim_{t\to\infty} T(x, t) = \frac{\rho_0 q_i}{k} \sum_{m=1}^{\infty} \frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2} + \frac{\rho_0 q_i}{k} \frac{C(\lambda_0)\phi_0(x)\cos(\lambda_0 c)}{\lambda_0^2}[1 - \exp(-\kappa\lambda_0^2 t)] \qquad \text{(A2)}$$

Then take the derivative of Eq. (A2) with respect to time

$$\frac{\partial T}{\partial t} = \frac{\rho_0 q_i \kappa C(\lambda_0)\phi_0(x)\cos(\lambda_0 c)}{k}[\exp(-\kappa\lambda_0^2 t)] \qquad \text{(A3)}$$

Then solve Poisson's equation for this condition

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho_0 q_i C(\lambda_0)\phi_0(x)\cos(\lambda_0 c)}{k}[\exp(-\kappa\lambda_0^2 t)] \qquad \text{(A4)}$$

The solution of this differential equation is

$$T(x, t) = \frac{\rho_0 q_i}{k\beta^2} C(\beta)\cosh[\beta(x - c)]\cosh(\beta c)\exp(\kappa\beta^2) + a_1 x + a_0 \qquad \text{(A5)}$$

Where a_1 and a_0 are constants of integration. Enforcing the boundary conditions in Eq. (3) and equating the result to Eq. (A2) yields the following relation:

$$\frac{\rho_0 q_i}{k} \sum_{m=1}^{\infty} \frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2} + \frac{\rho_0 q_i}{k} \frac{C(\lambda_0)\phi_0(x)\cos(\lambda_0 c)}{\lambda_0^2} = -\frac{\rho_0}{\alpha} \qquad \text{(A6)}$$

From which it follows that

$$T(x, t) = -\frac{\rho_0}{\alpha} - \frac{\rho_0 q_i}{k} \sum_{m=1}^{\infty} \frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2} \cdot \exp(-\kappa\lambda_m^2 t) + \frac{\rho_0 q_i}{k} \frac{C(\beta)\phi_0(x)\cosh(\beta c)}{\beta^2}\exp(\kappa\beta^2 t) \qquad \text{(A7)}$$

Equation (A7) converges rapidly and $\partial T/\partial x$ converges uniformly on the interval $0 < x < c$.

References

[1]Wieting, T. J., and Shriempf, J. T., "Infrared Absorbtances of Partially Ordered Alloys at Elevated Temperatures," *Journal of Applied Physics*, Vol. 47, No. 9, 1976, pp. 4009–4011.

[2]Gallant, D. J., "Temperature Dependence of the Optical Absorption in Metals," Univ. of Southern California, Physics Thesis, Los Angeles, CA., June 1980.

[3]Hartman, P., *Ordinary Differential Equations*, 2nd ed., Birkhauser, Boston, MA, 1982, pp. 337–344.

[4]Ölcer, N. Y., "On the Theory of Conductive Heat Transfer in Finite Regions," *International Journal of Heat and Mass Transfer*, Vol. 7, 1964, pp. 307–314

Mixing of Multiple Coflowing Confined Laminar Streams

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Introduction

THE improved design of chemical reactors, combustors, injectors, and other devices requiring the mixing of multiple streams, depends on our ability to obtain rapid and efficient diffusion of momentum, mass, and heat among these streams. In the traditional technology, the most frequently employed flow configuration for fluid mixing is the confined circular jet with or without a secondary coflowing stream. Recent requirements of improved performance, emphasizing robustness and efficiency, call for new approaches in the design of such rapid mixing equipment. In this note, it is proposed to study the possibility of using multiple streams to enhance the mixing process, thus resulting in more compact and less expensive devices.

The development of a circular jet with a coflow in a circular pipe has been studied for its significance in a number of engineering applications. The first systematic and comprehensive experimental study of the problem is that of Razinsky and Brighton.[1] In their work, pitot probes and hot-wire anemometry were used to study the effects of Reynolds number, velocity ratio (jet to secondary coflow velocity), and radius ratio (jet radius to pipe radius) on the flowfield. Other studies followed, including that of Suzuki et al.[2] which involved nonisothermal conditions, concentrating on different flow regimes. However, it was the study of Khodadadi and Vlachos[3] which provided a comprehensive set of measurements using the laser doppler anemometry (LDA) technique that led to a better understanding of the influence of large-scale organized structures on the flowfield, especially in the initial mixing region. In the same study a k-ε model of turbulence was employed to numerically predict the experimental data with limited success.

All studies up to now, however, involve the mixing of two coaxial streams (i.e., a central jet and a coflow around it). In the present work, the flow and heat transfer characteristics of systems with three coaxial laminar streams are numerically analyzed. The differences in the mixing process and the existence of separated flow regions resulting from the presence of the additional streams are determined. A second-order accurate, strongly implicit technique is employed for the solution of the governing fully elliptic Navier-Stokes equations. Results obtained for a range of the governing parameters are presented and demonstrate the increased mixing due to the presence of the third stream.

Governing Equations

The governing equations for the axisymmetric, steady, laminar, incompressible flow of a Newtonian fluid with constant properties in nondimensionalized form are

$$\nabla \cdot V = 0 \qquad (1)$$

$$(DV/Dt) = -\nabla p + (1/Re)\nabla^2 V \qquad (2)$$

$$(D\theta/Dt) = (1/Pe)\nabla^2\theta \qquad (3)$$

Received Jan. 27, 1992; revision received June 10, 1992; accepted for publication June 24, 1992.

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